UNIT - I
PROJECTILES

Definition:

1. The angle of projection is the angle that the direction in which the particle is initially projected makes with the horizontal plane through the point of projection
2. The velocity of projection is the velocity with which the particle is projected
3. The trajectory is the path which the particle describes
4. The range on a plane through the point of projection and the point where the trajectory meets that plane
5. The time of flight is the interval of time that elapse from the instant of projection till the instant when the particle again meets the horizontal plane through the point of projection.

The path of a projectile:

Let a particle be projected from O with a velocity u at an angle α to the horizon.

Take O as the origin, the horizontal and upward vertical through O as axes of $x \; and \; y$ respectively.

The initial velocity u can be split into two components, which are u cosα in the horizontal direction and using in the vertical direction

The horizontal component u cosα is constant throughout the motion as there is no horizontal acceleration.

The vertical component using α is subject to an acceleration g down wards. Let p (x , y) be the position of the particle at time t secs after projection .

Then x = horizontal distance described in t seconds

= (u cosα).t …………..①

Y= Vertical distance described in t sec

= (u sin α).t - $\frac{1}{2}gt^2$…………..②

From ① t= $\frac{x}{u\,cos\alpha}$ - $\frac{1}{2}g\frac{x^2}{u^2cos^2\alpha}$

$y = x\;tan\;\alpha - \frac{gx^2}{2u^2cos^2\alpha}$ ……………..③

=>$2u^2cos^2\alpha y = x\;tan\;\alpha . 2u^2cos^2\alpha - gx^2$

= $\frac{2u^2cos^2\alpha y}{g} = \frac{2u^2 - sin\;\alpha\;cos\;\alpha x}{g}$ - x^2

= >$\left(x^2 - \frac{2u^2 - \sin\alpha\cos\alpha\, x}{g}\right)^2 = \frac{2u^2\cos^2\alpha}{g}\ y$

= >$\left(x - \frac{u^2 - \sin\alpha\cos\alpha}{g}\right)^2 - \frac{u^4 - \sin^2\alpha\cos^2\alpha}{g^2} = \frac{-2u^2\cos^2\alpha}{g}\ y$

= $\left(x\frac{-u^2 - \cos^2\alpha}{g}\right)^2\left(y\frac{-u^2 - \sin^2\alpha}{2g}\right)$

Transfer the origin to the point

= $\left(\frac{u^2 - \sin\alpha\cos\alpha}{g}, \frac{u^2\sin^2\alpha}{2g}\right)$,

Then $x^2 = \frac{-2u^2 - \cos^2\alpha}{g}$, y …….④

④ is the equation of the paral with latusrectam$\frac{2u^2\cos^2\alpha}{g}$,

Whose axis is vertical and downwards and where vertex

1's the point =$\left(\frac{u^2 - \sin\alpha\cos\alpha}{g}, \frac{u^2\sin^2\alpha}{2g}\right)$

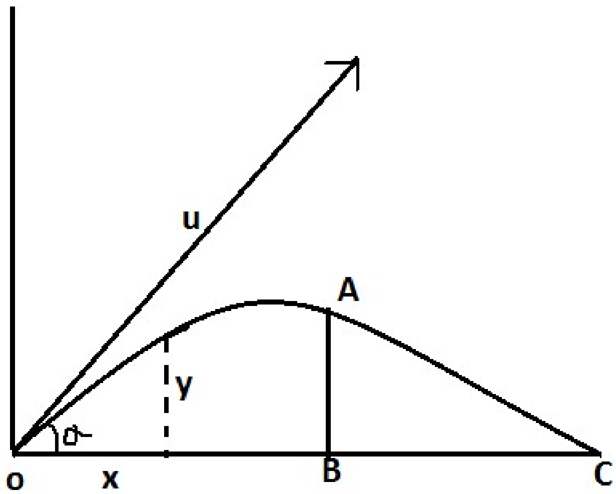
Latus rectum m = $\frac{2u^2\cos^2\alpha}{g}$

= $\frac{2}{g}(u\cos\alpha)^2$

= $\frac{2}{g}$ x square of the horizontal velocity.

Characteristic of the motion of a projectile:

1. Greatest height attained by a projectile:



At A, the highest point the particle will be moving only horizontally , having lost all of its vertical verlocity.

Let AB = h = the great at height reached.

Initial upward vertical velocity = u sin α ($v^2 = u^2 + 2as$)

The acceleration = -g

The final vertical velocity = 0

Hence 0 =(u sin $\alpha)^2$+2(-g)h.

= >$u^2\ si\ \ ^2\alpha = 2gh$

h= $\frac{u^2\ sin^2\ \alpha}{2g}$

i.e) the vertex of the parabola is two highest point of the path

2. Time taken to reach the greatest height:

Let T be the time from 0 to A .At this time initial velocity u sin α is reduced to zero.

Hence O =u sin αg $^{v=u+at}$

= > T = $\frac{\mathrm{u}\sin\alpha}{g}$

1. Time of flight:

When the particle arrives at c, the effective vertical distance it has described is zero.

∴=ut +$\frac{1}{2}at^2$

= > O = u sin α.t -$\frac{1}{2}gt^2$

= >t(u sin α-$\frac{1}{2}gt) = 0$

= > t= 0 or using -$\frac{1}{2}gt = 0$

= > t= $\frac{2u\ sin\alpha}{g}$

But t $\neq$0 = >t= $\frac{2u\ sin\alpha}{g}$

Time of flight = $\frac{2u\ sin\alpha}{g}$

Also t = 2T

2. At the time of flight , the horizontal velocity remains constant and is equal to $u\ cos\ \alpha$

Range = OC = horizontal distance described in time t.

$$= u\ cos\ \alpha.t$$

= $u\ cos\ \alpha.\frac{2u\ sin\alpha}{g}$

= $\frac{2u^2 sin\ \alpha\ cos\alpha}{g}$

R = $\frac{u^2 \sin 2\ \alpha}{g}$

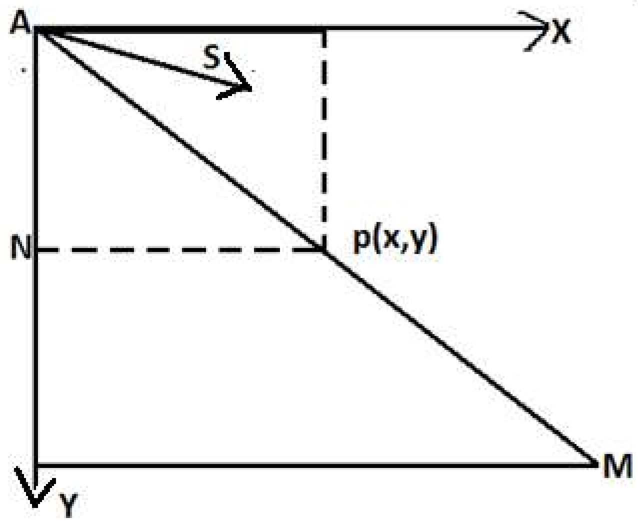
Also R= $\frac{2(u \cos\alpha)(\ \ sin\ \alpha\ )}{g}$

= $\frac{2UV}{g}$ where U & V are initial horizontal velocity & vertical velocity respectively.

Result:

A Particle is projected horizontally from a point at a certain height above the ground. Show that the path described by it is a parabola.

Proof:



Let a particle be projected horizontally with a velocity u from a point A at a height h above the ground level.

Let it strike the ground at M.

Take A as origin, the horizontal through A as x axis and the downward vertical through A as y axis

Let p(x, y) be the position of the particle at time t. As there is no horizontal acceleration, the horizontal velocity remains constant throughout the motion.

Now x = horizontal distance in time t

= ut ………………①

Y= vertical distance in time f

= $\frac{1}{2}gt^2$ …………….②

① => t= $\frac{x}{u}$

② = > y = $\frac{1}{2}g(x^2/_{u^2})$

Y= $\frac{.g}{2u^2}$ $\times 2$

= >x^2= $\frac{2u^2}{g}$ ………..③

∴Hence ③ represents a parabola with vertex at A and axis AN

Problems:

1. **If the greatest height attained by the particle is a quarter of its range on the horizontal plane through the point of projection , Find the angle of projection.**

Greatest height = $\frac{u^2 \sin^2 \alpha}{2g}$

Range = $\frac{2u^2 - \sin\alpha \cos\alpha}{g}$

Given height = $\frac{1}{4}$Range

=> $\frac{u^2 \sin^2 \alpha}{2g} = \frac{1}{4}\frac{2u^2 - \sin\alpha \cos\alpha}{g}$

= > sin α = cos α

$\boxed{\alpha = 45^o}$

2. **A Stone is thrown with a velocity of 39.2 mls at 30^o to the horizontal. Find at what times it will be at c height of 14.7 m**

Solution:

Given initial velocity = 39.2

$\alpha = 30^o$

y = vertical height =14.7

y= vertical distance at time t = usin α. t $- \frac{1}{2}gt^2$

14.7 =39.2 sin 30^o .t -$\frac{1}{2}$9.8 t^2

14.7 =39.2 $\frac{1}{2}$t -4.9 t^2

14.7= 19.6t - 4.9 t^2

147 = 196t - 49 t^2

3 = 4t - t^2

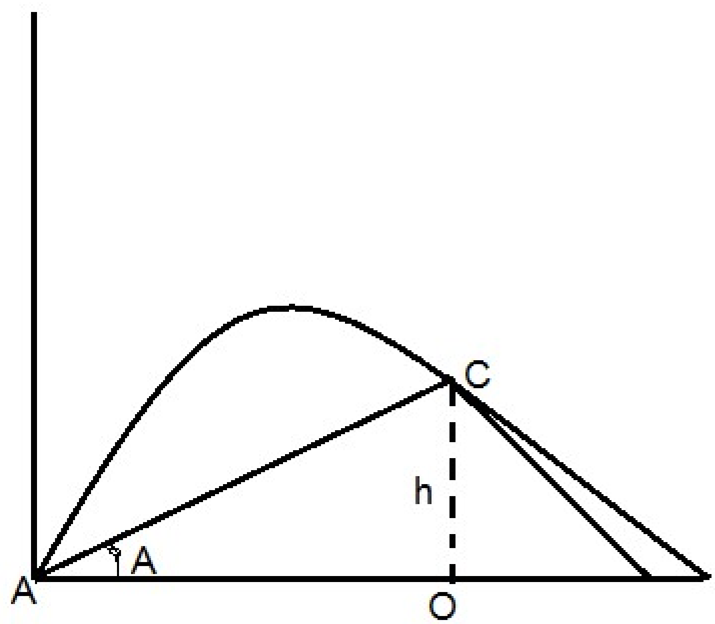
t^2 -4t + 3 = 0

t = 1 or t =3

Hence at time 1 sets and again at time 3 secs it will be at a height 14.7 m

3. A particle is thrown over a triangle from one end of a horizontal base and grazing the vertex falls on the other end of the base. If A, B are the base angles, and α the angle of projection. Show that tan α = tan A + tan B

Solution:



Let u be the initial velocity and α be two angle of projection and let t secs be the time

from A to C

Draw CD ⊥AB and let CD= h

considering vertical motion.

CD = h = u sin α . t $-\frac{1}{2}gt^2$

AD = horizontal distance

= u cosα. t

From ΔACD,

Tan A = $\frac{CD}{AD} = \frac{\text{u sin α .t} - \frac{1}{2}gt^2}{\text{u sin α .t}}$

In Δ CBD,

Tan B = $\frac{CD}{DB}$

Now DB = AB – AD

= $\frac{2u^2 - sin\ \alpha\ cos\ \alpha}{g} - u\ cos\ \alpha$

∴tanB = $\frac{\text{u sin α .t} - \frac{1}{2}gt^2}{\frac{2u^2 - sin\ \alpha\ cos\ \alpha}{g} - u\ cos\ \alpha}$

= $\frac{\frac{t}{2}(2\text{u sin α} - \text{gt})}{\frac{u\ cos\ \alpha}{g}((2\text{u sin α} - \text{gt})}$

tan B= $\frac{gt}{2u\ cos\ \text{α}}$

tan A + tan B= $\frac{\text{u sin α t} - \frac{1}{2}gt^2}{\text{u cos α .t}} + \frac{gt}{2u\ cos\ \text{α}}$

= $\frac{\text{2 u sin α .t} - gt^2 + gt^2}{\text{2u cos α .t}}$

tan A + tan B =tan α

Maximum horizontal range:

If u is two initial velocity and α is two angle of projection, the range 12 on the horizontal plane through the point of projection is given by

R = $\frac{u^2 \sin 2\ \alpha}{g}$

Since g being constant, for a given value u, the value of R is maximum , when $\sin 2\ \ \alpha$ is maximum

i.e) $\sin 2\ \ \alpha$ = 1

= > 2 α= 90^o

= >α= 45^o

Hence for a given velocity of projection, the horizontal range is a maximum when the particle is projected at on angle of 45^o to the horizontal.

When α= 40°, then R = $\frac{a^2}{g}$

Result:

For a given initial velocity of projection there are in general two possible directions of projections so α to obtain a horizontal range.

Proof:

Then k= $\frac{u^2 \, sin2 \, \alpha}{g}$

$\Rightarrow sin2 \, \alpha = \frac{gk}{u^2}$ ……………①

Since u, k are given & g is a constant R. H. S of ① is a known positive quantity

If gk<u^2 we can determine an acute angle θ

Whose sin is equal to$\frac{gk}{u^2}$

①$sin2 \, \alpha$ = sin θ

$\Rightarrow 2 \, \alpha = \theta$

$\alpha = {}^{\theta}/_{2}$

Also sin (180 - $\theta) = si$

⇒sin (180 - $\theta) = \sin 2 \;\alpha$

⇒ 180 - $\theta \, = 2\alpha$

⇒ $\alpha = 90 - {}^{\theta}/_{2}$

Hence we find that there are two values of α and so two directions of projection , each giving the same range t . Let $\alpha_1 \;_{and}\; \alpha_2$ be those two values of α.

Then $\alpha_{1=} \; {}^{\theta}/_{2}$, $\quad \alpha_2 = 90 - {}^{\theta}/_{2}$

$\Rightarrow \alpha_{1\,+}\, \alpha_2 = 90°$

$\Rightarrow \alpha_{1\,+}\, \alpha_2 = 45.145$

$\Rightarrow \alpha_2 - 45 = 45 - \alpha_1$ ………………②

As θ<90°, α_1< 45 &$\alpha_2 > 45°$

But 45^o is two angle of projection to get maximum horizontal range with the same initial velocity.

Hence ② shows that the two directions $\alpha_1 \ _{and}\ \alpha_2$are equally inclined to the direction of maximum range.

Note:

If u^2<gk , the R. H.S $\frac{gk}{u^2}$> 1 and so we cannot find the real value for α

i.e) There is no angle of projection to get a range greater than $\frac{u^2}{g}$which is really the maximum range.

Problem:

1) **If h & n' be the greatest heights in the two paths of a projectile with a given velocity for a given ran R.**

Prove that R=$\sqrt[4]{hh'}$

Solution:

Let α and α ′ be two angles of projection

Then h = $\frac{u^2 \sin^2 \alpha}{2g}$

h' = $\frac{u^2 \sin^2 \alpha}{2g}$

hh' = $\frac{u^4 \, si \ \ ^2\alpha \, sin^2\alpha 1}{4g^2}$

Also we know that
$\alpha + \alpha\,' = 90$

= >α ′= 90 - α

hh' = $\frac{u^4 \, sin^2\alpha \, si \ \ ^2(\,90-\alpha\,)}{4g^2}$

hh' = $\frac{u^4 \, sin^2\alpha \, sin^2\alpha}{4g^2}$

$$\Rightarrow \sqrt{hh'} \frac{u^2 \sin\alpha \cos\alpha}{2g}$$

$\Rightarrow \sqrt[4]{hh'} = \frac{2u^2 \sin\alpha \cos\alpha}{g}$

$= \frac{u^2 \sin 2\alpha}{g}$

$= \sqrt[4]{hh'} = R$

2. **Prove that in any trajectory over a horizontal plans , the horizontal range is a maximum when it is equal to four times the greatest heights**

Solution:

Horizontal range R = $\frac{u^2 \sin^2 \alpha}{g}$

Greatest height h = $\frac{u^2 \sin^2 \alpha}{2g}$

Given R = 4h

$\frac{u^2 \sin^2 \alpha}{g} = \frac{4\,u^2 \sin^2 \alpha}{2g}$

2sin α cos α = $2\sin^2 \alpha$

⇒cosα = sin α

⇒ tan α= 1

$\boxed{\alpha = 45^\circ}$

The horizontal range is maximum

3. **A bomb resting range is maximum ground explods sending fragment in all directions with a velocity of 98m/s . What is the greatest distance from the bomb at which a fragment can fall?**

Solution:

Given u = 98m/s

For $\alpha = 45^\circ$, the horizontal distance is maximum,

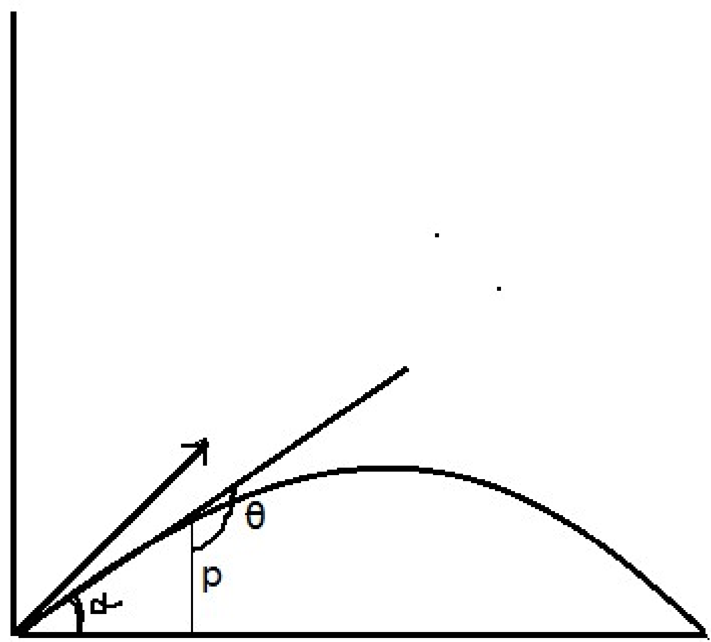
∴maximum distance = $\frac{u^2}{g}$

$= \frac{98 \times 98}{9.8}$

= 980 m

To find the velocity of the projectile in magnitude and direction at the end of time t

Let a particle be projected from O with a velocity a at an angle α to the horizon . After time t , let it be at p and v be its velocity inclined at an angle θ to the



horizontal

Horizontal component at p= v cosθ

But throughout horizontal component is same

⇒ u cos$\alpha v\ cos$ θ …………①

$v\ sin$ θ = u sin α-gt ………….②

Squaring & adding we get

$v^2 cos^2\theta + v^2 sin^2\theta = u^2 cos^2\alpha + u^2 sin^2\alpha + g^2t^2 - 2gt\, usin\,\alpha$

$$\Rightarrow v^2 = u^2 + 2usin\,\alpha.\text{gt} + g^2t^2$$

⇒ v = $\sqrt{u^2 - 2u\sin\alpha\,\text{gt} + g^2t^2}$ ………….③

$\frac{②}{①}$ ⇒ tan θ = $\frac{\text{u}\sin\alpha - \text{gt}}{\text{u}\cos\alpha}$

Equation ③ &② gives the velocity at p in magnitude and direction.

Note:

(i) If t < $\frac{u \sin \alpha}{g}$

Then $u \sin \alpha - gt > 0$

⇒ $\tan \theta > 0$

⇒ $\theta > 0$

If t > $\frac{u \sin \alpha}{g}$

Then $u \sin \alpha - gt < 0$

⇒ $\tan \theta < 0$

⇒ $\theta < 0$

If t = $\frac{u \sin \alpha}{g}$

Then $u \sin \alpha - gt = 0$

⇒ $\tan \theta = 0$

⇒ $\theta = 0$

Hence at the highest point A , the direction of velocity is horizontal

(ii) Put t= $\frac{2u \sin \alpha}{g}$, its time of flight

Then v = $\sqrt{u^2 - 2sin\alpha \,.\, g^2 \frac{2u \, sin \, \alpha}{g} + g^2 \frac{4u^2 \, sin^2\alpha}{g^2}}$

= $\sqrt{u^2 - 4u^2 sin^2\alpha + 4u^2 \, sin^2\alpha}$

= V' = u

$\tan \theta = \frac{2sin\alpha \,. g^2 \frac{2u \sin \alpha}{g}}{\mathrm{u} \cos \alpha}$

= $\frac{\mathrm{u} \sin \alpha}{\mathrm{u} \cos \alpha}$

Tan θ= 1- $\tan\alpha = \tan(-\alpha)$

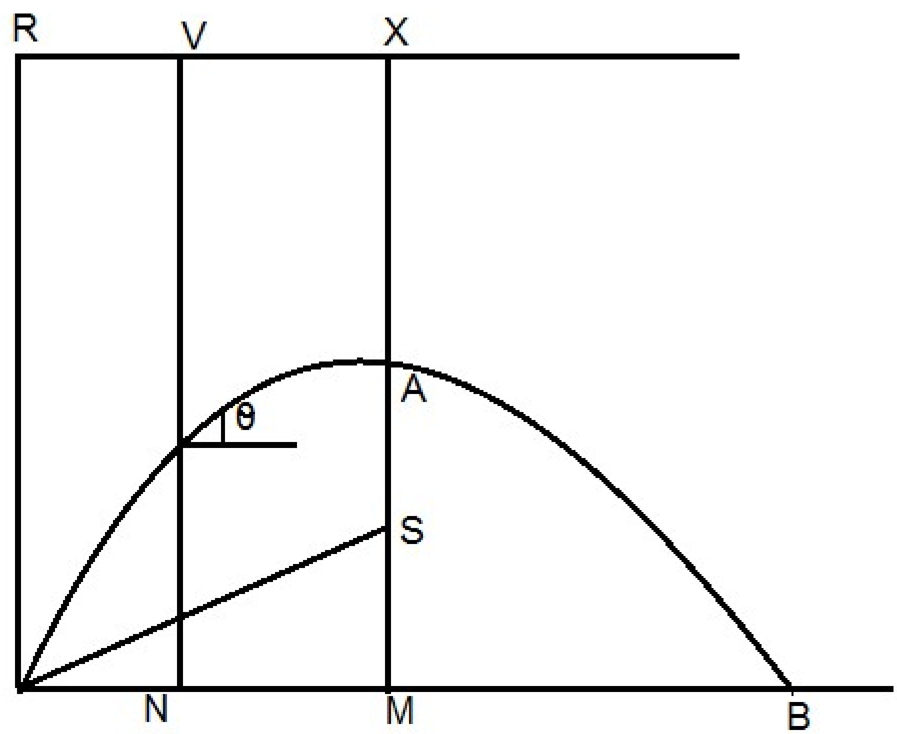
$$\Rightarrow \theta = -\alpha$$

Hence the particle strikes the horizontal plane downwards with the same velocity as the initial velocity and at the same angle as that with which it was projected.

Result:

The velocity at any point P of a projectile is equal in magnitude to the velocity acquired in falling freely from the direction to the point.

Proof:



If v is the velocity at an angle θ to the horizontal. When the particle is at p .at t end t sec , we have

$v^2 = u^2 - 2usin\alpha\,.\mathrm{gt} + g^2t^2$ ……….①

Let L be the point vertically above xxxxxx directory of the path. If v is two velocity acquired by a particle which falls freely under gravity from L to P $v^2 = u^2 + 2gs$, then

v^2 =2gLp ………②

Let s be the focus, A the vertex and $x$the foot of the directions

Ax = As = $\frac{1}{4}$x latus rectum

= $\frac{1}{4}$x $\frac{2u^2 cos^2 \alpha}{g}$

= $\frac{u^2 cos^2 \alpha}{2g}$

x m = A$x + AM$

= $\frac{u^2 cos^2 \alpha}{2g} + \frac{u^2 sin^2 \alpha}{2g} = \frac{u^2}{2g} = Ln$

PN = Vertical distance travelled in t secs

= $u \, sin \, \alpha$ t $- \frac{1}{2}$ gt^2

LP = LN – PN

= XM – PN

= $\frac{u^2}{2g} - u \, sin \, \alpha$ t $+ \frac{1}{2}$ gt^2

v^2= 2g $\left[\frac{u^2}{2g} - u \, sin\alpha \text{ t} + gt^2\right]$

= $u^2 - 2gu \sin \alpha$ t $+ gt^2$

= v^2

UNIT - II
COLLISION OF ELASTIC BODIES

Definition: (Restitution)

The internal force which acts, when a body lends to recover its original shape after deformation or compression is called the force of restitution.

The property which causes a solid body to recover its shape is called elasticity.

If a body does not bend to recover its shape , it will cause no force of restitution and such a body is said to be inelastic.

Remark:

Suppose a ball is dropped from any height h upon a hard floor. It strikes the floor with a velocity u = $\sqrt{2gh}$ and makes an impact , soon it rebound and moves vertically upwards with a velocity v.

The height h_1 to which is given by $h_1 = \frac{r^2}{2g}$

i.e) v = $\sqrt{2gh}$

we have $h_1 < h$. So $v<u$

Definition:

1. If v = u , the velocity with which the ball leaves the floor is the same as that with which it strikes it. In this case , the ball is said to be inelastic.
2. When a body completely regains its shape after a collision , it is said to be perfectly elastic.
3. If it does not come to its original shape , it is said to be perfectly inelastic
4. Two bodies are said to impinge directly when the direction of motion of each before impact is along the common normal at the point where they touch.
5. Two bodies are said to impinge obliquely, if the direction of motion body is not along the common normal

Newton's Experimental Law:

When two bodies, impinge directly, their relative velocity after impact beds a constant ratio to their relative velocity before impact and is in opposite direction.

When two bodies, impinge obliquely, their relative velocity after impact bears a constant ratio to their relative velocity before impact and is in same direction.

i.e) $\frac{v_2 - v_1}{u_2 - u_1} = -e$

Note:

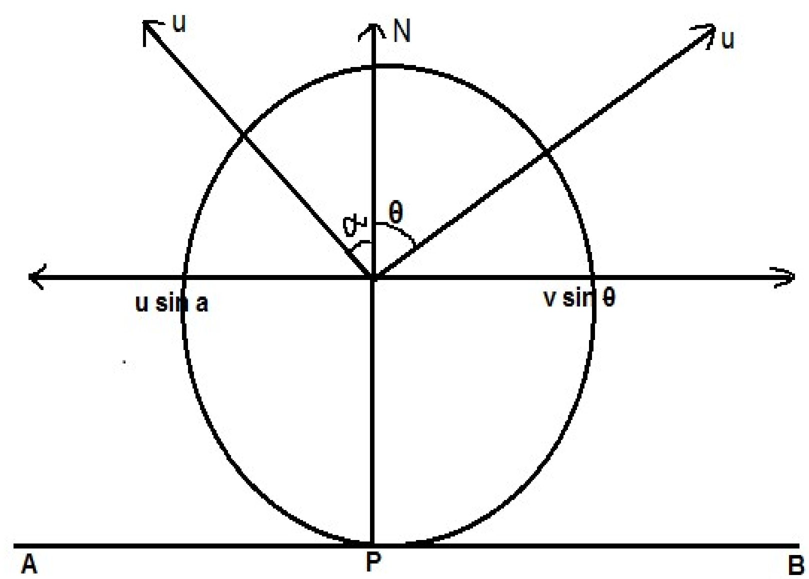
1. O < e < 1
2. For two glass balls ex
3. For ivory e =0.8
4. For lead e = 0.2
5. For two balls, one of lead and one iron e = 0.13
6. For e = 0, bodies are inelastic.
7. For e =1 , bodies are elastic

Impact of a smooth Sphere on a fixed smooth plane:

Let AB be the plane and p the point at which the sphere strikes it.

The common normal at p is the vertical line at p pausing through the centre of the sphere

Let it be PC. This is the line of impact. Let the velocity of the sphere before impact be u at an angle α with CP and V its velocity after impact at an angle Q win CN as shown in the figure



Since the plane and the sphere are smooth the only force acting during impact is the impulsive along the common normal.

There is no parallel force to the plane during impact. Hence the velocity of the sphere resolved in a direction parallel to the plane is unaltered by the impact.

Hence v sin θ= u sin α ……………①

By newton's law,

$$\frac{\boldsymbol{v_2 - v_1}}{\boldsymbol{u_2 - u_1}} = -e$$

$$\Rightarrow \frac{v\,cos\theta - 0}{-u\cos\alpha - 0} = -e$$

$\Rightarrow v\,cos\theta = -\mathrm{e}(u\cos\alpha)$

$v\,cos\theta = \mathrm{e}u\cos\alpha$ ……….②

$$V^2 sin^2\theta + V^2 cos^2\theta = u^2 sin^2\alpha + e^2 u^2 cos^2\alpha$$

$$\Rightarrow V^2 = u^2(si\ \ ^2\alpha + e^2 cos^2\alpha)$$

⇒ V = u $\sqrt{sin^2\alpha + e^2 cos^2\alpha}$……………③

$\frac{②}{①} \Rightarrow \frac{v\,cos\theta}{v\,sin\theta} = \frac{\mathrm{e}u\cos}{u\sin\ \alpha}$

⇒ cot θ = e cot α ………..④

③ < ④ gives the magnitude and direction of motion after impact

Corollary 1:

If e = 1 θ = α & v = u

Sphere impinges on a fixed smooth plane, its velocity is not altered by impact and the angle of reflection is equal to the angle of incidence

Corollary 2:

If e = 0,

Cot θ = 0

θ = 90°

Hence the inelastic sphere slides along the plane with velocity u sin α

Corollary 3 :

If the impact is direct

We have $\alpha = 0$

Then v sin θ= 0 θ = 0

Also v = eu

Hence if an elastic sphere strikes a plane normally with velocity u, it will rebound in the same direction with velocity eu

Corollary 4 :

The impulse of the pressure on the plane is equal and opposite to the impulse of the pressure on the sphere . The impulse I on the sphere is measured by the momentum of the sphere along the common normal .

I = mv cosθ – (-mu cos α)

= m (v cosθ + u cos α)

= m (eucosα + u cos α)

= mu cosα(1 + e)

Corollary 5 :

Loss of k . E due to impact

$= \frac{1}{2}mu^2 - \frac{1}{2}mv^2$

$= \frac{1}{2}mu^2 - \frac{1}{2}mu^2(sin^2\alpha + e^2cos^2\alpha)$

$= \frac{1}{2}mu^2[\,1 - \ sin^2\alpha + e^2cos^2\alpha]$

$= \frac{1}{2}mu^2(cos^2\alpha - e^2cos^2\alpha)$

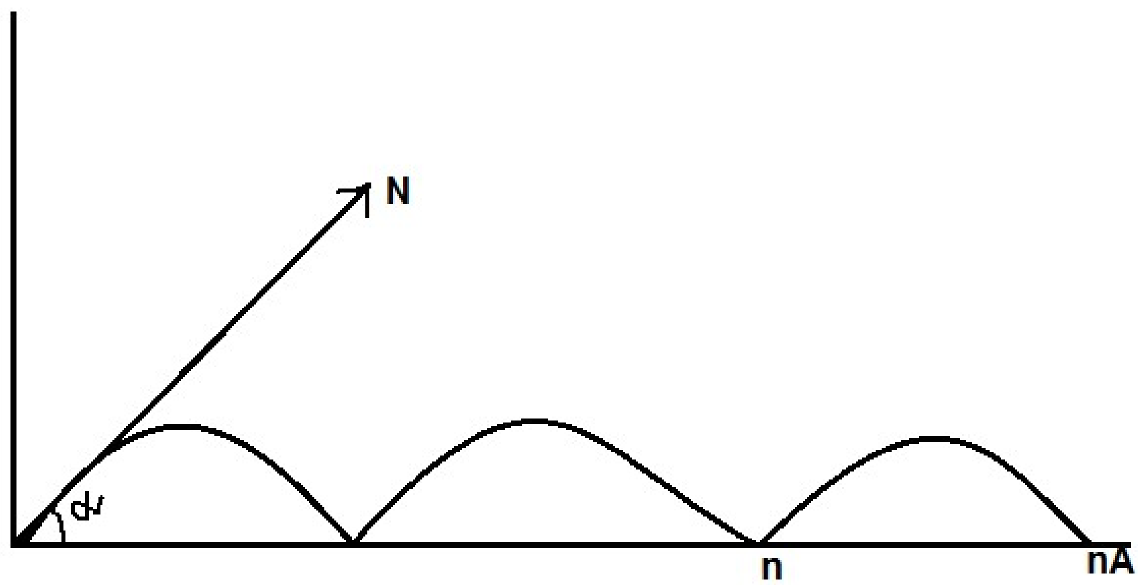
$= \frac{1}{2}mu^2cos^2\alpha(\,\mathrm{H} - e^2)$

If the sphere is perfectly elastic, then e = 1 , and the loss of k. E .s zero .

Problems:

1. **A particle is projected from a point on a smooth horizontal plane with a velocity v at an elevation α and continues to rebound show that the range between nm and (n+1) m, impacts is $\frac{V^2 e^n \sin 2\alpha}{g}$; the coefficient of restitution**

Solution:



The ball at first describes parabola and on striking the plane it rebounds and describes another parabola etc., after first impact

The components are v sin (90−α)& ev cos (90 - α)

i.e) v cosα ev sin α

After second impact the components are $v\ cos\ \alpha$, $e^2 v \sin\alpha$

In general , the components after nth impact are

$v\ cos\ \alpha$, $e^2 v \sin\alpha$

Let t_1 , t_2 , t_3 , etc be the times for the successive trajectories .

Then $t_1 = \frac{2v \sin}{g}$

Range between

= ut + $\frac{1}{2}at^2$

= v cosα. tn

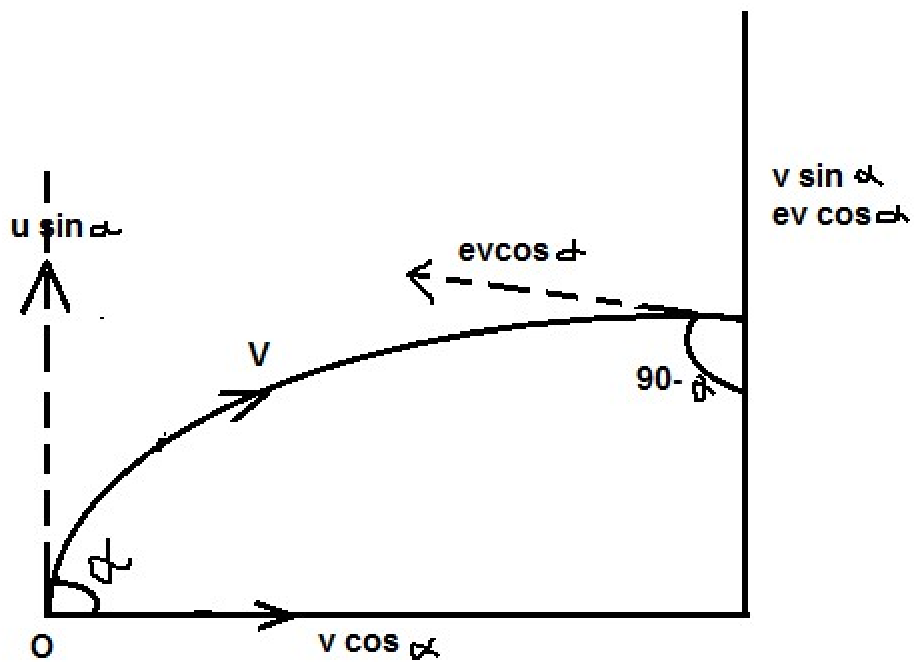
= v cosα. $\frac{2ve^{n}\sin\alpha}{g}$

= $\frac{v^{2}e^{n}\sin\alpha}{g}$

2. **An elastic sphere is projected from a given point o with given velocity v at an inclination α to the horizontal and after hitting a smooth vertical wall at a distance d from o returns to o .**

   Prove that d = $\frac{v^2 \sin 2\alpha}{g}\frac{e}{1+e}$ where e is the coefficient of restitution.

   Solution:



Let o be the point of projection and the particle strikes the wall at A win time t_4 .

Also it returns to o after time t_2

Considering horizontal distance for OA:

S = ut+ $\frac{1}{2}at^2$

d = v cosαt_1 + 0

= > t_1 = $\frac{d}{v\cos\alpha}$ ………… ①

After impact the components are $v\ sin\ \alpha$

Similarly considering the horizontal distance for AO,

d = $e\ v\ cos\ \alpha . t_2$

$=> t_2 = \frac{d}{e\ v \cos \alpha}$

Since the particle returns to bit place after time $t_1 + t_2$ the vertical distance after time $t_1 + t_2$ is zero.

Considering perpendicular distance

$S = ut + \frac{1}{2}at^2$

$\Rightarrow O = v \sin \alpha (t_1 + t_2) - \frac{1}{2}g (t_1 + t_2)^2$

$\Rightarrow \frac{1}{2}g (t_1 + t_2) = v \sin \alpha$

$\Rightarrow g \left(\frac{d}{v \cos \alpha} + \frac{d}{ev \cos \alpha}\right) = 2 v \sin\alpha$

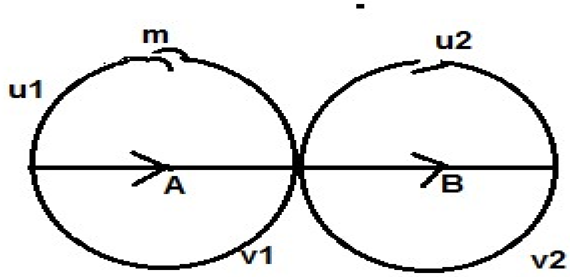
$\Rightarrow gd\left(\frac{e+1}{ev \cos \alpha}\right) = 2 v \sin\alpha$

$\Rightarrow d = \frac{2e . v^2 \sin \alpha \cos \alpha}{g (e+1)}$

$d = \frac{ev^2 \sin 2\alpha}{(e+1)g}$

Direct impact of two smooth spheres:

A smooth sphere of mass m_1, impinges directly with velocity u, on another smooth sphere of mass m_2 , morning in the same direction with velocity u_2 if the coefficient of restitution is e, to find their velocities after impact



Let AB be the line of impact

By Newton's Law,

$(v_2 - v_1) = -e\,(u_2 - u_1)$ …….①

$= -eu_2 + eu_1$

By the principle of conversation of momentum, the total momentum along normal before impact is equal to the total momentum along normal after impact.

Ie) $m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2$

① x m1 $-m_1v_1 + m_1v_2 = m_1eu_1 = em_1u_2$

$(m_1+m_2)v_2 = m_1u_1\,(1+e)$

$V_2 = \frac{m_1u_1(1+e)+m_2-e\quad)u_2}{m_1m_2}$ …………………③

① $=> v_1 = v_2 + eu_2 - eu_1$

$V1 = \frac{m_1u_1(1+e)+m_2-em\,)u_2}{m_1+m_2} + eu_2 - eu_1 = \frac{m_2u_2(1+e)+u_1(m_1-em_2\,)}{m_1+m_2} + em_2$ ……….④

Equation ③ & ④ gives the velocities of the sphere after impact

UNIT – III
SIMPLE HARMONIC MOTION

Introduction:

Suppose one end of an elastic string is tied to a fixed point and a heavy particle is attached to the other end. If the particle is disturbed vertically from its position of equilibrium, it is found that it oscillates to an fro about this position. Clearly the particle cannot be moving under constant acceleration. It is found that it has an acceleration which is always directed towards the equilibrium position and varies in magnitude as the distance of the partial from that position. This kind of motion occurs frequently in nature and since it is of the type which produces all musical holes, it is called simple Harmonic Motion.

Examples:

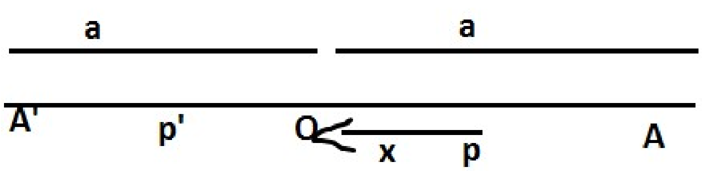
1. The oscillations of a simple pendulam
2. The transverse vibrations of a plucked violin string

Definition:

When a particle moves in a straight line so that its acceleration is always directed towards a fixed point in the line and proportional to the distance from that point , its motion is called simple harmonic motion.

Simple Harmonic Motion in a straight line:

Let O be a fixed point on the straight line. A ' OA on which a particle is having Simple Harmonic motion



Let p be the position of the particle at tim t and OP = x

By definition of Simple Harmonic motion

Accerlation.α .distance = x

i.e)Accerlation = - μx

{Since we consider the motion on the direction ox }

$\frac{d^2x}{dt^2}$ = - μx ……… ①

Similarly the acceleration is in positive quantity.

Equation ① is the fundamental differential equation representing a Simple Harmonic motion

To Solve ①

$\frac{d^2x}{dt^2}$ = - μx

$$\Rightarrow \frac{d}{dt}\left(\frac{dx}{dt}\right) = -\mu x$$

$$\Rightarrow \frac{dv}{dt} = -\mu x$$

$$\Rightarrow \frac{dv}{dx}.\frac{dx}{dt} = -\mu x$$

$$\Rightarrow v.\frac{dv}{dx} - \mu x$$

$$\Rightarrow vdv = -\mu x \ dx$$

Integrating we get

$\frac{v^2}{2} = \frac{-\mu \times 2}{2} + c$ ……………②

Initially the particle start from rest at the point A where OA = a

At this time r = a , v= 0

②⇒ O = $\frac{-\mathcal{M}\ a2}{2} + c$

⇒ c = $\frac{\mathcal{M}\ a2}{2}$

② ⇒ $\frac{v^2}{2} = \frac{-\mu \times 2}{2} + \frac{\mu a2}{2}$

3. If the displacement of a moving point out any time be given by an equation of the form

r = c coswt + b sin wt . Show that the motion is a simple harmonic motion is a simple harmonic motion. If a = 3, b = 4 , w=2 determine the period amplitude , maximum velocity and maximum acceleration of the motion.

Solution:

Given x = coswt + b sin wt ……..①

$\frac{dx}{dt}$ = a (-sinwt) .w + b cos wt .w

= -aw sinwt + bwcoswt

$\frac{d^2x}{dt^2}$= -aw^2 coswt –bw^{2}sinwt

= -w^2 (a coswt + bsinwt)

= -w^2x = $-$ μx

Hence x is a simple harmonic motion with

$\mathcal{M}$= w^2 => w = $\sqrt{-\mu}$

Period = $\frac{2\pi}{\sqrt{\boldsymbol{\mu}}}$ = $\frac{2\pi}{w}$

Given a =3 , b= 4 , w = 2

Period = π

Amplitude is the greatest value

$x$is maximum when $\frac{du}{dt} = 0$

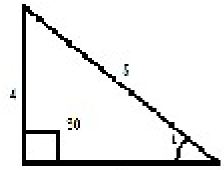
And $\frac{d^2x}{dt^2}$= negative.

$\frac{dx}{dt}$ = 0 => awsinwt + bw coswt

⇒ aw sinwt = bwcoswt

⇒ 3 sin 2t = 4cos 2t

90tan 2t = $\frac{4}{3}$



Then sin2t = $\frac{4}{5}$

Cos 2t = $\frac{3}{5}$

① => $x = 3\cos 2t + 4\sin 2t$

$= 3 \cdot \frac{3}{5} + 4 \cdot \frac{4}{5} = 5$

Amplitude =5

Maximum acceleration = μ x Amptitude

$= w^2 \times 5 = 4 \times 5$

= 20

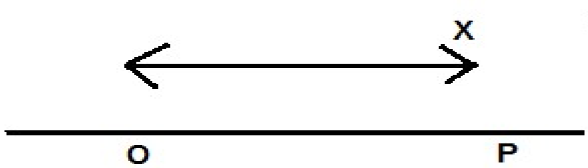
max .velocity = $\sqrt{\mu}$ x amptitute

= 2 x 5

= 10

4. **A horizontal shelf moves vertically with SHM whose, complete period is one second . Find the greatest amplitude in centimeters it can have, so that an object resting on the shelf may always remain in contact.**

Solution:



Let m be the mass of the particle , and let o be the centre of SHM and R be the position of the particle with op = r

The forces acting on the particle are (i) Its weights acting downwards

(ii) Normal reaction r acting upwards

Resultant force = Mass – R

= mg - R

Now force = m x a

⇒ mg – R = m . Acceleration

⇒ Acceleration = $\frac{\text{mg} - \text{R}}{m}$

By simple harmonic motion

$\frac{d^2x}{dt^2}$ = - μx .along op.

Considering along po,

$\frac{d^2x}{dt^2}$ = μx

i.e) Acceleration = μx …………..①

from ① & ② we get

$\frac{\mathrm{mg} - \mathrm{R}}{m} = \mu x$

$\mathrm{mg} - \mathrm{R} = \mu x\, m$

R = mg – mμr

Since R≥ 0

M (g - μr) ≥ 0 …………….. ③

Also period = 1 sec

$\frac{2\pi}{\sqrt{\mu}} = 1$

$\Rightarrow \sqrt{\mu} = 2\pi \Rightarrow \mu = 4\pi^2$

③ $\Rightarrow$ g - $4\pi^2$r ≤ g

$\Rightarrow$ g ≥ $4\pi^2$ r

$\Rightarrow$ $4\pi^2$r ≤ g

r ≤ $\frac{g}{4\pi^2}$

Maximum value of r = $\frac{g}{4\pi^2}$

$= \frac{9.8}{4 \times 3.14}$ m

$= \frac{9800}{4 \times 3.14}$ cms

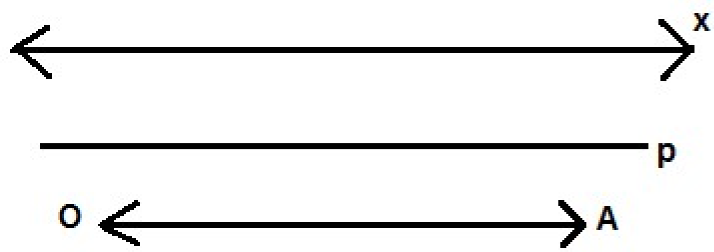
5. **A particle p of mass m moves in a straight line ox under a force m μdirected towards a point A which moves in a straight line ox with constant acceleration . Show that the motion of P is SHM , a period $\frac{2\pi}{\sqrt{\mu}}$ behind A .**

Solution:

Let at time t the particle be at p where op = x and A be such that OA = y .The equation of motion at p is

$\frac{d^2x}{dt^2}$ = -μPA= $-\mu(x - y)$

The equation of motion of A is $\frac{d^2y}{dt^2} = \alpha$



$\frac{d^2x}{dt^2} - \frac{d^2y}{dt^2} = -\mu(x - y) - \alpha$

$= -\mu\left(x - y + \frac{\alpha}{\mu}\right)$

Let z = $x - y + \frac{\alpha}{\mu}$

$\frac{dz}{dt} = \frac{dx}{dt} - \frac{dy}{dt}$

$\frac{d^2z}{dt^2} = \frac{d^2x}{dt^2} - \frac{d^2y}{dt^2}$

① => $\frac{d^2z}{dt^2} = -\mu z$

∴This is a simple harmonic motion with period $\frac{2\pi}{\sqrt{\mu}}$

=> z = $x - y + \frac{\alpha}{\mu}$

= AP $+\frac{\alpha}{\mu}$

Let B be any point between O & A such that

BA = $\frac{\alpha}{\mu}$

∴ **z =** AP +BA = BP

i.e) z denotes the displacement of p measured from B .

Hence the motion at p is SHM about B a moving centre is always at a distance $\frac{\alpha}{\mu}$ behind A.

6. **A particle of mass m is oscillating in a straight line about a centre of force O , towards which when at a distance r, the force mn^2r and 'a' is the amplitude of the oscillation when at a distance $\frac{a\sqrt{3}}{2}$ from o, the particle receives a bloco in the direction of motion which generates a velocity na . If this velocity be always from O, Show that the new amplitude is $a\sqrt{3}$.**

Solution:

Given force =$m.n^2r$

Acceleration = n^2

∴ $\mu = n^2$

Velocity $v^2 = \mu^2 (a^2 - x^2)$

When x = $\frac{a\sqrt{3}}{2}$

$v^2 = n^2(a^2 - \frac{3a^2}{4})$

since the additional velocity given to the particle is away from O , in the direction of previous motion. We take positive sign.

∴ **v** = $\frac{na}{2}$

The total velocity = $\frac{na}{2} + na = \frac{3na}{2}$

The subsequence motion as again SHM.

If v is the velocity at any distance x from o ,& A is new amptitude

$V^2 = n^2 (A^2 - x^2)$ ………………………….①

If x = $\frac{a\sqrt{3}}{2}$, v = $\frac{3na}{2}$

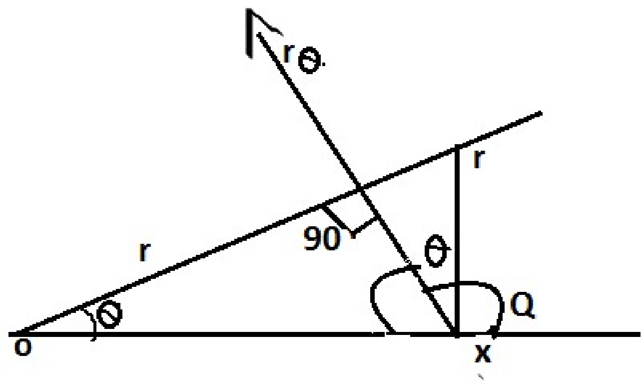
① $\Rightarrow \frac{9n^2a^2}{2} = n^2\left(A^2 - \frac{9a^2}{4}\right)$

$\Rightarrow A^2 = \frac{9a^2}{4} + \frac{3a^2}{4} = 3a^2$

$\Rightarrow A = a\sqrt{3}$

UNIT - IV
MOTION UNDER THE ACTION OF CENTRAL FORCE

Velocity and acceleration in polar coordinates:



Let p be the position of a moving particle at time t.

Taking O as the pole and ox as the initial line , let the polar coordinates of p be (r, θ)

$\overrightarrow{op}$ = r is the position of the vector of p.

∴velocity of p = $\frac{d}{dt}$(r)

Since $\vec{r}$ has modules r and amplitude θ

$\frac{d}{dt}$(r) have two components

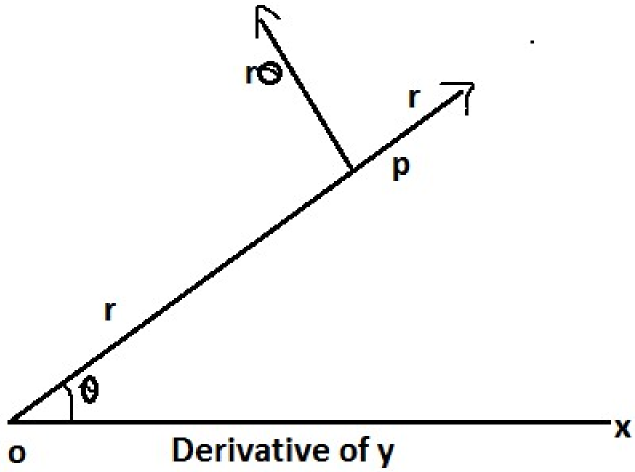
(i) r along op , which is called radial components of v
(ii) r θ perpendicular to op in the direction in which θ increased which is called transverse component of v

The radial component of v is a vector with modules $r_{ij}$and and amplitude θ

∴ The derivative of r will have components

(i) r along op in the direction r increases
(ii) r θ perpendicular to op in the direction in which θ increases

This is shows by the following figure

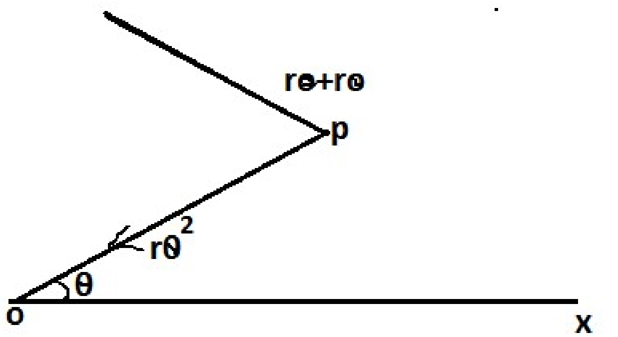


The transverse component of v is a vector with modules r θ and amplitude $\phi = {}^{\pi}/_{2} + \theta$

∴ The derivative of r θ eill have components.

(i) $\frac{d}{dt}(r\,\theta) = r\,\theta + \theta r$ along the line of r θ i.e $\perp$ op

(ii) $r\,\theta\,\frac{d}{dt}\,({}^{\pi}/_{2} + \theta) = r\,\theta^2$ in the direction of perpendicular to the line of



∴ The components of acceleration are $r - r\,\theta^2$ in the direction op and $r\,\theta + 2r\,\theta$ in the perpendicular direction.

Note:

Now $\frac{1}{r}\frac{d}{dt}(\,s) = \frac{1}{r} r^2\,\theta + 2rr\,\theta)$

$= r\,\theta + 2r\theta$

∴Acceleration perpendicular to op is also = $\frac{1}{r}\frac{d}{dt}(r^2\ \theta)$

	Magnitude	Direction	Sence
Radial component of velocity	$\dot{r}$	Along the radius vector	In the direct in which r increases
Transverse component of velocity	r θ	Perpendicular to the radius vector	In the direction in which θincreases
Radial component of acceleration	$r- r\ \theta^2$	A long the radius vector	In the direction in which r increases
Transverse component of acceleration	$\frac{1}{r}\frac{d}{dt}(r^2\ \theta)$	Perpendicular to the radius vector	In the direction in which θ increases

Corollary:

Suppose the particle p is describing a circle of radius ‘ a ‘.Then r =a through out the motion

Hence $\ddot{r} = 0$

The radical acceleration

$= \ddot{r} - r\dot{\theta}^2$

$= 0 - a\dot{\theta}^2 = -a\dot{\theta}^2$

Transversal acceleration

$= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$

$= \frac{1}{a}\frac{d}{dt}(a^2\dot{\theta})$

$= a\frac{d}{dt}(\dot{\theta}) = a\dot{\theta}$

∴ Hence for a particle describing a circle of radio a, the acceleration at any point p has the components a $\ddot{\theta}$ along the tangent at p and $a\dot{\theta}^2$ along the radios to the centre.

Corollary 2:

Magnitude of the resultant velocity of p

= $\sqrt{\dot{r}^2 + (r\dot{\theta})^2}$

Equiangular spiral:

Equiangular spiral is a curve which has the important properly that the tangent at any point p on it makes a constant angle with the radius vector op.

Let op &θQ be two consecutive radii vectors such that the included angle

POQ = Δθ

Let OP = r, OQ = r + Δr

Draw QL ⊥ OP

Cos Δθ = $\frac{OL}{OQ}$ = > OL = OQ cos Δt

OL = (r+ Δr) cos Δθ

Since Δθ is small, cos Δθ= 1

∴OL= r + Δr

PL = OL – OP = r + Δr- r = ΔL

Sin Δθ= $\frac{OL}{OQ}$ = > QL = OQ sin Δθ

QL = (r+ Δr) sin Δθ

Since Δθ is small, sinΔθ = Δθ

∴QL = (r + Δr) Δθ

= r . Δθ+ Δr . Δθ

= r Δθ (approximately)

tan∠QPL = $\frac{QL}{PL}$

= $\frac{r\Delta\theta}{\Delta r}$

In the limit as Δr and Δθ both o, the point Q, tends to coincide with P.

The chord QP becomes in the limiting position the tangent at P.

Let Q be the angle made by the tangent at P with OP

Then $\phi = \lim_{Q \to p} \angle QPL$

$\tan \phi = \lim_{Q \to p} tan \ \angle QPL$

$= \lim_{\Delta r \to 0} r . \frac{\Delta\theta}{\Delta r}$

$\tan \phi = r . \frac{d\theta}{dr}$

It gives the angle between the radios vector and the tangent

For equiangular spiral, at any point p on it, the angle ϕ is constant

Let p = α

Then tanϕ= tan α

i.e) $r \frac{d\theta}{dr} = \tan \alpha$

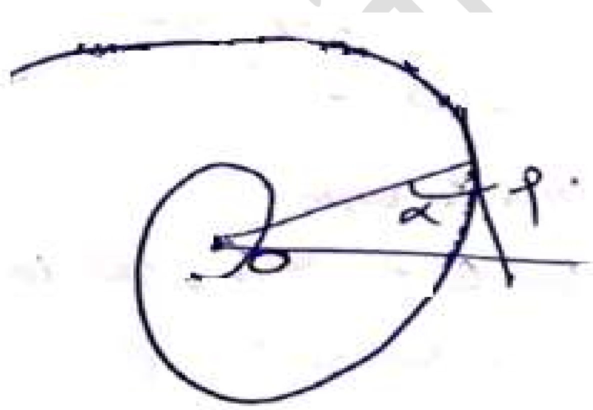
$\Rightarrow \frac{dr}{r} = \frac{1}{tan\ \alpha,} d\,\theta$

Integrating we get, $\log r = \theta.\cot \alpha + c$

$\Rightarrow r = ae^{\theta \cot} \alpha$

Where $a=e^c$

This is the polar equation to the equiangular spiral the curve is



Problems:

1. The velocities of a particle along and perpendicular to a radius vector from a fixed origin are λr^2 and $\mu \theta^2$

Where μ and λ are constants show that the equation to the path of the particle is $\frac{\lambda}{\theta}$+c = $\frac{\mu}{2r^2}$ where c is a constant . Show also that the acceleration along and perpendicular to the radio vector are

$2\lambda^2 r^3 - \frac{\mu^2\theta^4}{r}$ and

$\mu\,(\lambda r\theta^2 + \frac{2\mu\theta^3}{r})$

Solution:

Given radial velocity

r = λ r^2

$\Rightarrow \frac{dr}{dt} = \lambda\, r^2$ ....①

Perpendicular velocity

rθ = $\mu\theta^2$

$\Rightarrow r\frac{d\theta}{dt} = \mu\theta^2$.............................②

$\frac{dr}{dt} / r\frac{d\theta}{dt} = \frac{\lambda\, r^2}{\mu\theta^2}$

$\Rightarrow \frac{dr}{r\, d\theta} = \frac{\lambda\, r^2}{\mu\theta^2}$

$\Rightarrow \frac{dr}{r^3} = \frac{\lambda d\theta}{\mu\theta^2}$

$\Rightarrow \frac{\mu\, dr}{r^3} = \lambda\frac{d\theta}{\theta^2}$

$\Rightarrow \frac{\mu\, r^{-3+1}}{-3+1} = \frac{\lambda\theta^{-2+1}}{-2+1} + c$

$\Rightarrow \frac{\mu}{-2r^2} = \frac{\lambda}{-\theta} + c$

$\Rightarrow \frac{\mu}{2r^2} = \frac{\lambda}{\theta} - c$

$\frac{\mu}{2r^2} = \frac{\lambda}{\theta} + c$

Where c is a constant .

Acceleration along radios

$= \ddot{r} - r\,\dot{\theta}^2$

$= \frac{dr^2}{dt^2} - r\left(\frac{dv}{dt}\right)^2$ .... ③

Diff ① we get

$\frac{d^2r}{dt^2} = \lambda 2r.\frac{dr}{dt}$

$= 2\,\lambda r.\,\lambda r^2$

$\frac{d^2r}{dt^2} = 2\,\lambda^2\,r^3$

③ Radial Acceleration

$= 2\,\lambda^2\,r^3 - r.\left(\frac{\mu\,\theta^2}{r}\right)^2$

$= 2\,\lambda^2\,r^3 - r\,\frac{\mu^2\theta^4}{r^2}$

$= 2\,\lambda^2\,r^3 - \frac{\mu^2\theta^4}{r}$

Transverse Acceleration

$= \frac{1}{r}\frac{d}{dt}(r^2\,\theta)$

$= \frac{1}{r}\frac{d}{dt}(r^2\frac{d\theta}{dt})$

$= \frac{1}{r}\frac{d}{dt}(r^2\,\frac{\mu\,\theta^2}{r})$

$= \frac{1}{r}\frac{d}{dt}(\mu\,\theta^2)$

$= \frac{\mu}{r}[\,r\,2\,\theta.\frac{d\theta}{dt} + \theta^2\,\frac{dr}{dt}]$

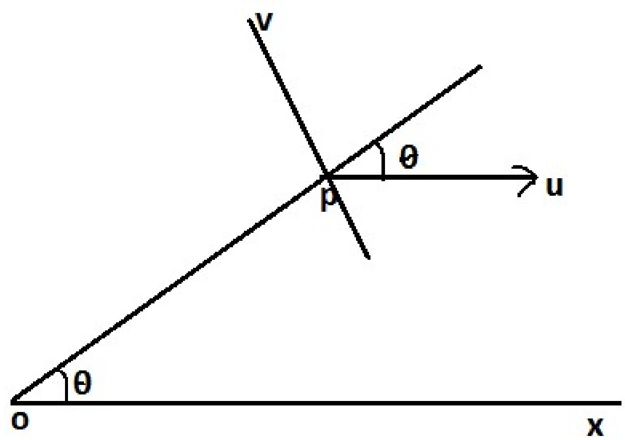
$= \frac{\mu}{r}[\,r\,2\,\theta.\frac{\mu\,\theta^2}{r} + \theta^2\,\lambda r^2\,]$

$= \frac{\mu}{r}[\,2\,\mu\,\theta^3 + \lambda\,\theta^2\,r^2\,]$

$= \mu\,[\lambda r\theta^2 + \frac{2\,\mu\,\theta 3}{r}\,]$

2. Show that the path of a point p which possesses two constant velocities u and v , the first of which is in a fixed direction and the second of which is perpendicular to the radius OP .Drawn from a fixed point O , is a conic. Whose focus is 0 and whose eccentricity is $\frac{u}{v}$

Solution:



Take 0 as the pole and the line ox parallel to the given direction as the initial line p has two velocities u parallel to ox and v perpendicular to op. Resolving the velocities along op,

R = u cosθ = $\frac{dr}{dt}$ = u cos …………………①

Resolving velocities perpendicular to op, we have,

r θ = v –u sin θ

r.$\frac{d\theta}{dt}$ = v – u sin θ ……………………………②

$\frac{①}{②} \Rightarrow \frac{dr\ /dt}{rd\theta/\mathrm{dt}} = \frac{u\cos\theta}{v-u\ sin\theta}$

$\Rightarrow \frac{dr}{rd\theta} = \frac{u\cos\theta}{v-u\ sin\theta}$

$\Rightarrow \frac{dr}{r} = \frac{d\ \theta\ \ u\cos\theta}{v-u\ sin\theta}$

$\frac{dr}{r} = \frac{-\ d\ (\mathrm{v}-u\sin\theta)}{v-u\ sin\theta}$

Integrating we get,

$\log r$ = - $\log(v - us \quad \theta\ ) + \log c$

$\Rightarrow$ r = $\frac{c}{v - u\ sin\theta}$

$\Rightarrow$ r (v- u sin θ)= c

$\Rightarrow$ $\frac{c}{r}$ = v - u $sin\theta$

$\Rightarrow$ $\frac{1}{r}\left({}^{c}/_{r}\right)$ = 1 - $\frac{u}{v}$ sin θ

= 1+ $\frac{u}{v}\cos(90 + \theta\ )$

This is of the form

$\frac{\lambda}{r}$ = 1+ e cos $(\theta + \alpha)$

Comparing l = ${}^{c}/_{v}$, e = ${}^{u}/_{v}$

α = 90^{o}

∴The palm is a conic whose focus at O , semi – lotus rectum is ${}^{c}/_{v}$, accentricity ${}^{u}/_{v}$and whose major axis I perpendicular to the initial line.

3. A Point p describes a curve with constant velocity and its cirgular velocity about a given fixed point o varies inversely as the distance from o. Show that the curve is an equiangular spiral whose pole is O , and that the acceleration of the point is along the normal at p and varies inversely as op.

Solution:

Taking o as the pole, let p be (r, θ)

Resultant velocity of p

= $\sqrt{r^2 +\ r^2\theta\ ^2}$ = constant

Take $\sqrt{r^2 +\ r^2\theta\ ^2}$ = k

Angular velocity about O = θ

= $\frac{\lambda}{r}$ …………………②

② $\Rightarrow r^2 +\ r^2\theta\ ^2$= k^2

$\Rightarrow r^2 +\ r^2\frac{\lambda^2}{r^2}$ = k^2

$\Rightarrow r^2 + \lambda^2 =$ k^2

$\Rightarrow r^2 = k^2 - \lambda^2$

$\Rightarrow r = \sqrt{k^2 \lambda^2}$ ……………….③

$\frac{dr}{dt} = \sqrt{k^2 \lambda^2}$

$\frac{d\theta/dt}{dr/dt} = \frac{\lambda/r}{\sqrt{k^{2-}\lambda^2}}$

$$\Rightarrow \frac{d\theta}{dr} = \frac{\lambda}{r\sqrt{k^{2-}\lambda^2}}$$

$\Rightarrow \sqrt{k^2 - \lambda^2}\ d\theta = \frac{\lambda\, d\, r}{r}$

$\Rightarrow \frac{d\, r}{r} = \frac{\sqrt{k^2 - \lambda^2}}{\lambda}\ d\theta$

Integrating we get,

$\log r = \frac{\sqrt{k^2 - \lambda^2}}{\lambda}\theta + c$

$\Rightarrow r = e\ \frac{\sqrt{k^2 - \lambda^2}}{\lambda}\ \theta.\ e^c$

Put $e^c = a$, $\frac{\sqrt{k^2 - \lambda^2}}{\lambda} = \cot\alpha$

Then $r = ae^{\theta \cot\alpha}$

∴ The path is in equation angular spiral . Whose pole is differentiating ③

$\ddot{r} = 0$

Radial acceleration = $\ddot{r} - r\dot{\theta}^2$

$= r - \frac{\lambda^2}{r^2} = \frac{-\lambda^2}{r}$

∴ The radial acceleration at p is along po,

Transverse acceleration = $\frac{1}{r}\frac{d}{dt}\ (r^2\ \theta)$

$= \frac{1}{r}\frac{d}{dt}\ (r^2 \frac{\lambda}{r})$

$= \frac{1}{r}\frac{d}{dt}\ (\ \lambda r\ )$

$= \frac{1}{r}\ \lambda r$

$= \frac{\lambda}{r}\sqrt{k^2 - \lambda^2}$

Resultant acceleration of p

$= \sqrt{\left(\frac{-\lambda^2}{r}\right)^2 + \left(\frac{\lambda}{r}\sqrt{k^2-\lambda^2}\right)^2}$

$= \sqrt{\frac{\lambda^4}{r^2} + \frac{\lambda^2}{r^2}(k^2-\lambda^2)}$

$= \frac{\lambda}{r}\sqrt{\lambda^2 + k^2 - \lambda^2}$

$= \frac{k\lambda}{r}$

∴ The resultant acceleration varies inversely as r i.e) as op let this acceleration be along PN making an angle with PO .

tanβ $= \frac{\textit{Component perpendicular to po}}{\textit{component along po}}$

$= \frac{\frac{\lambda}{r}\sqrt{k^2-\lambda^2}}{\frac{\lambda^2}{r}}$

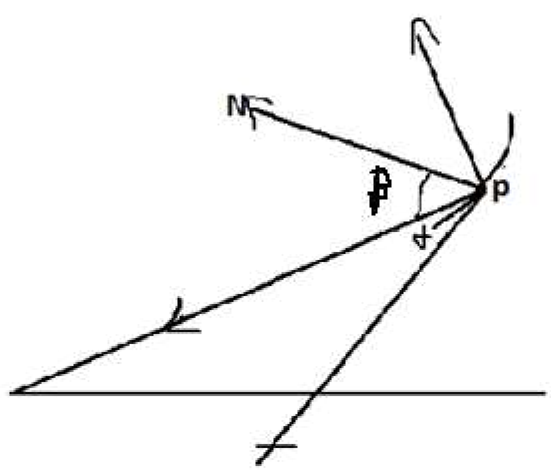
tanβ = $\frac{\sqrt{k^2-\lambda^2}}{\lambda}$ = cot α $= \tan(90 - \alpha)$

⇒ β = 90 -- α

⇒ $\alpha + \beta = 90^o$

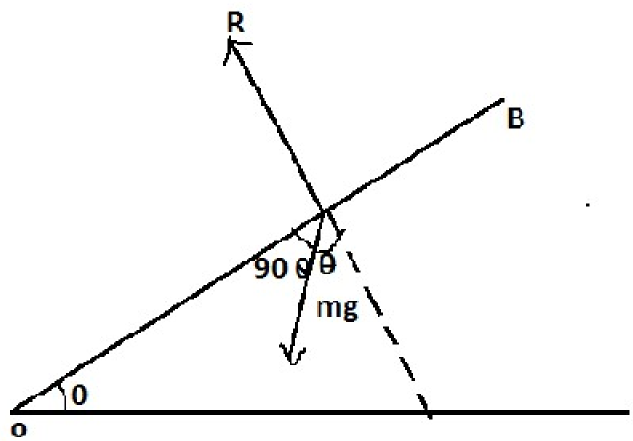
Hence ∟NPT = 90^o where PT is the tangent at P.

Hence PN is the normal at P



1. A Smooth straight thin tube evolves with uniform angular velocity w in a vertical plane about one extremity which is fixed , if at zero time the cube be horizontal and particle inside it be at a distance a from the fixed end and be moving with velocity v along the tube , show that the distance at time t is a cash wt + $\left(\frac{v}{w} - \frac{g}{2w^2}\right)$ sin wt+ $\frac{g}{2w^2}$ sin wt

Solution:



Let at time t, p be the position of the particle of mass m on the tube OB.

The forces acting at p and

(i) Its weight mg vertically downwards
(ii) Normal reaction R perpendicular to OB

Let p be (r, θ)

Angular velocity = θ = $\frac{d\theta}{dt} = w(given)$

Integrating we get,

Θ = wt +A , A constant

Initially when t=0 ,θ= 0 ⇒ A = 0

∴ θ = wt

Resolving along the radius vector OB

m ($\ddot{r}$ –r $\dot{\theta}^2$) = -mg cos (90 – 0)

⇒ $\ddot{r}$ –r $\dot{\theta}^2$ = -g sin θ

$\Rightarrow \ddot{r}$ –r w^2 = -g sin wt

$\Rightarrow \frac{d^2r}{dt^2}$ - w^2 – r= -g sin wt

$\Rightarrow$ (D^2- w^2) r = -gsinwt ……………①

Where P = $\frac{d}{dt}$

A.E.B D^2 - w^2 = 0

$\Rightarrow D^2 = w^2$

$\Rightarrow D = \pm w$

C.F = Ae^{wt}+ Be^{-wt}

P . I = $\frac{1}{D^2 - w^2}$ $(-g \sin wt)$

= $\frac{1}{-w^2 - w^2}$ $(-g \sin wt)$

= $\frac{g \sin w}{2\,w^2}$

∴ The general solution of ① is

r = Ae^{wt}+ Be^{-wt} + $\frac{g \sin w}{2\,w^2}$ ………………②

when t =0 , r = a &$\dot{r}$= v

put t =0 & r = a in ②

$\Rightarrow$ a = Ae^o+Be^o+$\frac{g \sin o}{2\,w^2}$

a = A + B ……………③

Diff ② with respect to t

$\dot{r}$ = WAe^{wt} - WBe^{-wt} + $\frac{g\ \ w \cos\ wt}{2\,w^2}$

$\dot{r}$ = WAe^{wt} - WBe^{-wt} + $\frac{g\ \ \cos\ wt}{2w}$

Put t = 0 &$\dot{r}$ =v

$\Rightarrow$ v =wAe^o - wBe^o + $\frac{g\ \ \cos\ 0}{2w}$

$\Rightarrow$v = WA –WB + $\frac{g}{2w}$

$\Rightarrow \frac{v}{w} = A - B + \frac{g}{2w^2}$

$\Rightarrow A - B = \frac{v}{w} - \frac{g}{2w^2}$ ………………. ④

③ + ④ $=> 2A = a + \frac{v}{w} - \frac{g}{2w^2}$

$A = \frac{a}{2} + \frac{v}{2w} - \frac{g}{4w^2}$

③ - ④ $= 2B = a - \frac{v}{w} + \frac{g}{2w^2}$

$\Rightarrow B = \frac{a}{2} - \frac{v}{2w} + \frac{g}{4w^2}$

The solution is $r = \left(\frac{a}{2} + \frac{v}{2w} - \frac{g}{4w^2}\right)e^{wt} + \left(\frac{a}{2} - \frac{v}{2w} + \frac{g}{4w^2}\right)e^{-wt} + \frac{g\sin}{2\,w^2}$

$= a\left(\frac{e^{wt}e^{-w}}{2}\right) + \left(\frac{v}{2w}\frac{-g}{4w^2}\right) x \left(\frac{e^{wt}-e^{-w}}{2}\right) + \frac{g\sin}{2\,w^2} = a \text{ coshwt} + \left(\frac{v}{2w} - \frac{g}{4w^2}\right)\sin hw$

UNIT - V
DIFFERENTIAL EQUATION OF CENTRAL ORBITS

A Particle moves in a plane with an acceleration which is always directed to a fixed point O in the plane, to obtain the differential equation of its path:

Take O as the pole and a fixed line through O as the initial line.

Let p (r , θ) be the polar coordinates of the particle at time t and m be its mass.

Also let p be the magnitude of the , central acceleration along po.

The equation of motion of the particle are

m ($\ddot{r}$ –r w^2) = -mp …………………..①

&m. $\frac{1}{r}\frac{d}{dt}(r^2\,\theta)$ = 0 ………………..②

Equation ② follows from the fact that as there is no force at right angles to op, the transverse component of the acceleration is zero throughout the motion

② ⇒ $\frac{d}{dt}(r^2\,\theta)$ = 0

⇒ r^2 θ= Constant = h

Put u = $\frac{1}{r}$ ⇒ r = $\frac{1}{u}$

③ ⇒ $\frac{1}{u^2}$ = h

Now $\dot{r}$ = $\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right)$

= $\frac{-1}{u^2}\frac{du}{dt}$

= $\frac{-1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt}$

= $\frac{-1}{u^2}\frac{du}{d\theta}$. θ

= $\frac{1}{-u^2}\frac{du}{d\theta}$ .hu^2

$\dot{r}$ = -h $\frac{du}{d\theta}$

$\ddot{r}$ = $\frac{d}{dt}\left[(h)\frac{du}{d\theta}\right]$

$= -h \frac{d}{dt}\left(\frac{du}{d\theta}\right)$

$= -h \frac{d}{d\theta}\left(\frac{du}{d\theta}\right) . \frac{d\theta}{dt}$

$= -h \frac{d^2u}{d\theta^2} . \theta$

$= -h \frac{d^2u}{d\theta^2} . (hu^2)$

$= -h^2u^2\frac{d^2u}{d\theta^2}$

$D \Rightarrow \ddot{r} - r\ \dot{\theta}^2 = -p$

$\Rightarrow h^2u^2\frac{d^2u}{d\theta^2} - r\ h^2u^4 = -p$

$\Rightarrow h^2u^2\frac{d^2u}{d\theta^2} + \frac{1}{u}\ h^2u^4 = p$

$\Rightarrow h^2u^2\ [\frac{d^2u}{d\theta^2} + u\ ] = p$

$\Rightarrow u + \frac{d^2u}{d\theta^2} = \frac{p}{\mathrm{h2u2}}$

This is the equation of the central orbit , in polar coordinates

Note :

If the central force is a repulsive one in a particular problem , the sign of p must be changed.

Pedal Equation of the central orbit:

In certain curves the relation between the perpendicular from the pole on the tangent and the radius vector is very simple such a relation is called the pedal equation or the (p, r) equation to the curve

The (p,r) equation to a central orbit as follows

We have $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$ ……………①

Difference between sides of ① with respect to θ

$-2p^3\frac{dp}{d\theta} = 2u\ \frac{du}{d\theta} + 2\frac{du}{d\theta}\frac{du}{d\theta}$

$\Rightarrow \frac{1}{p^3}\frac{dp}{d\theta}$ = u $\frac{du}{d\theta} + \frac{du}{d\theta}\frac{d^2u}{d\theta^2}$

= $\frac{du}{d\theta}\left(u + \frac{d^2u}{d\theta^2}\right)$

But the differential equation in polar is $u + \frac{d^2u}{d\theta^2} = \frac{p}{h^2u^2}$

$\Rightarrow \frac{-1}{p^3}\frac{dp}{d\theta} = \frac{du}{d\theta}\frac{p}{h^2u^2}$

$\Rightarrow \frac{-1}{p^3}\, dp = \frac{p}{h^2u^2}\, du$

$\Rightarrow \frac{-1}{p^3}\, dp = \frac{p}{h^2} r^2 d\left(\frac{1}{r}\right)$

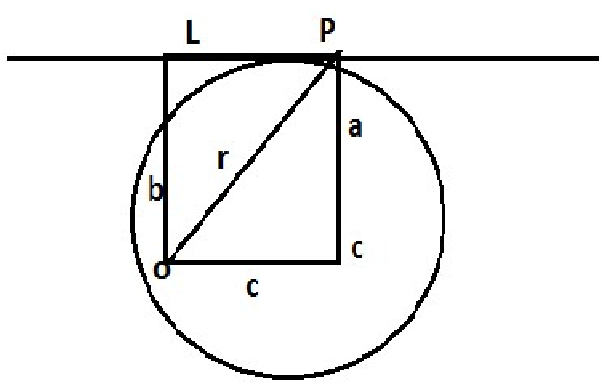
$\Rightarrow \frac{-1}{p^3}\, dp = \frac{pr^2}{h^2}\left(-\frac{1}{r}\right)$- dr

$\Rightarrow \frac{dp}{p^3} = \frac{p\, dr}{h^2}$

$\Rightarrow \frac{h^2}{p^3}\frac{dp}{dr} = p$

This is the (p ,r) equation or pedal , equation to the central orbit.

Pedal Equation of circle pole at any point:



Let c be the centre and a be the radius O the pole where OC = C

Let p be any point on the circle and OL be the perpendicular from O on the tangent at p.

OP = r & OL = p

Δ OPC

$C^2 = r^2 + a^2 - 2ra \cos \llcorner OPC$

$= r^2 + a^2 - 2ra . \frac{p}{r}$

$= r^2 + a^2 - 2ap.$

This is the pedal equation of the circle for a general position of the pole.

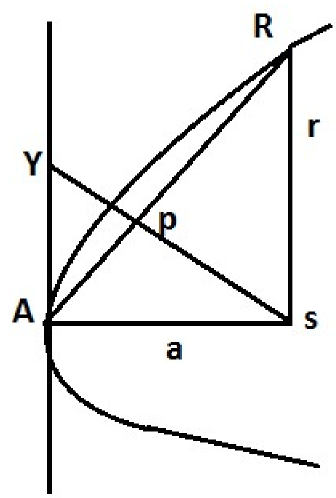
When c = a

i.e) the pole O is on the circumference

$a^2 = r^2 + a2 - 2ap$

$\Rightarrow r^2 = 2ap$

Pedal Equation of parabola Pole at focus:



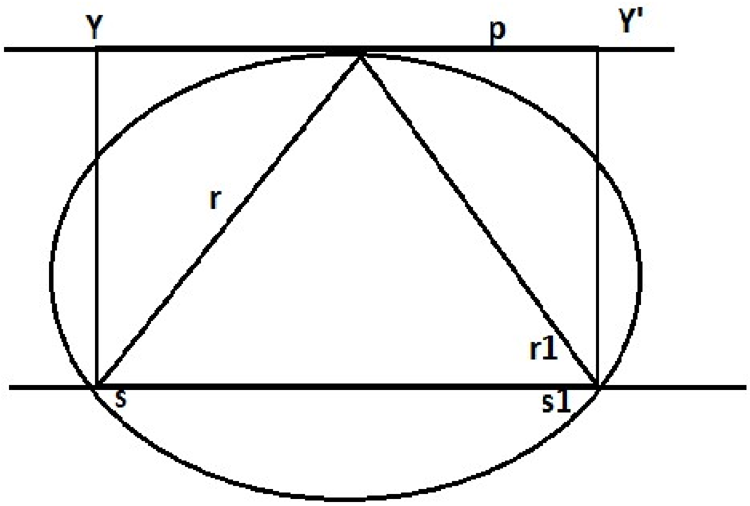
To get the (p, r) equation to a parabola, we assume the geometrical property that if the tangent at p meets the tangent at the vertex A in Y and S is the focus, then SY is perpendicular to PY and the trangles SAY and SYP aresimilar.

Hence $\frac{SA}{SY} = \frac{SY}{SP}$

$\Rightarrow \frac{a}{p} = \frac{p}{r}$

$$\Rightarrow p^2 = ar$$

Pedal Equation of ellipse or Hyperbola – Pole at focus:



Let s and s' be the focus of the ellipse and SY ,S' Y' be the perpendiculars to the tangent at p

Taking S as the pole,

Let SP = r , S' P' = r'

SY = p , S'Y' = P'

Let a and b be the semi axes. To find the (p, r) equation , we assume the following geometrical properties of the ellipse:

(i) SP + S'P = 2a
 = >r +r' = 2a
(ii) SY. SY' = b^2
 = > PP' = b^2
(iii) The tangent at p is equally inclined to the focal distance so that SPY and S'PY' are similar triangles.

∴we have $\frac{p}{r} = \frac{p\prime}{r\prime}$

$\frac{b^2}{p^2} = \frac{pp\prime}{p^2} = \frac{p\prime}{p}$

$\frac{b^2}{p^2} = \frac{r\prime}{r}$

= $\frac{2a-r}{r}$

$\frac{b^2}{p^2} = \frac{2a}{r} - 1$ is the (p,r) equation to the ellipse

Similarly , the (p , r) equation of the branch of the hyperbola nearest to the focus is $\frac{b^2}{p^2} = \frac{2a}{r} + 1$

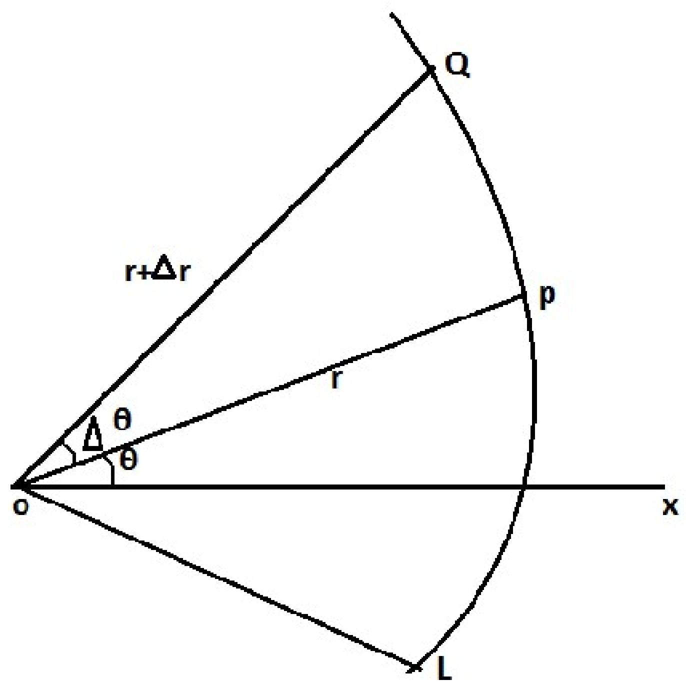
Pedal equation of Equiangular Spiral:

P = r sin ϕ be any curve.

In equiangular spiral ϕ = costant

= > p = kr is the pedal equation to the spiral

Velocities in a central orbit:



Let at time t the particle be at p (r ,θ) and at time t + Δt , let it be at Q (r + Δr , θ + Δθ)

Sectorial area OPQ described by the radius vector 90

= Area of Δ OPQ nearly,

= ½ OP OQ sin ∟POQ

= ½ r (r + $\Delta r) sin\, \Delta\, \theta$

= ½ r (r + $\Delta r)\, \Delta\, \theta$ (since $\Delta\, \theta$ is small)

= ½ $r^2 \Delta\, \theta$+ r Δ r$\Delta\, \theta$

= ½ $r^2 \Delta\, \theta$(approximately)

The rate of description of the area traced out by the radius vector joining the particle to a fixed point is called the areal velocity of the particle.

In the central orbit, areal velocity of p

= $\lim_{\Delta t \to \mathbf{0}} \frac{1}{2} r^2 \frac{\Delta\theta}{\Delta t}$

= ½ $r^2 \frac{d\theta}{dt}$

= ½ $r^2\, \theta$

= ½ h (since$r^2\, \theta = \text{constant} = \text{h})$

∴ Areal velocity is constant

⇒Equal areas are described by the radius vector in equal times.

Let Δ s be the length of the arc PQ

Draw OL perpendicular to PQ sectorial area = Δ POQ nearly

As Δt → $\mathbf{0}$, Q tends to coincide with P along the curve and the chord QP becomes the tangent at P.

Length PQ = Δs nearly & OL becomes the perpendicular from O on the tangent at p .

Let OL = P

Areal velocity = Lt ½ $\frac{\Delta s}{\Delta t}$. p

= ½ p $\frac{ds}{dt}$ = ½ pv

As $\frac{ds}{dt}$ is the rate of describing s and so is the linear velocity of p.

∴ ½ h = ½ pv

⇒ h = pv

$\Rightarrow$ v = h/p

Hence linear velocity varies inversely as OP.

Problems:

1. **Find the law of force towards the pole under which the curve $r^n = a^n \cos n\theta$ can be described**

Solution:

Given $r^n = a^n \cos n\theta$ ………………①

Put r = 1/ u

① $\Rightarrow \frac{1}{u^n} = a^n \cos n\theta$

$$\Rightarrow u^n a^n = \cos n \ \ \theta = 1$$

Taking log we get,

n log u + n log a + log cos n θ = 0

Difference in ② with respect to θwe get

$n. \frac{1}{u}\frac{du}{d\theta} + \frac{1}{\cos n\,\theta}(- \sin n\ \theta)\ n = 0$

$\Rightarrow \frac{n}{u}\frac{du}{d\theta} = + n \tan n\ \theta$

$\Rightarrow \frac{du}{d\theta} = u \tan n\ \theta$

Difference with respect of θ

$\frac{d^2u}{d\theta^2} = u \sec^2 n\ \theta . n\ + \tan n\ \theta . \frac{du}{d\theta}$

$= u n \sec^2 n\ \theta + u \tan^2 n\ \theta$

The equation of central orbit is , $U + \frac{d^2u}{d\theta^2} = p / h^2u^2$

$\frac{p}{h^2u^2} = U + u n \sec^2 n\ \theta + u \tan^2 n\ \theta$

$= u (1 + \tan^2 n\ \theta) + u n \sec^2 n\ \theta$

$= u \sec^2 n\ \theta + u n \sec^2 n\ \theta$

= u (1 + n)

= u (1 + n) u^{2n} . a^{2n}

$\frac{p}{h^2u^2}$ = u^{2n+1} (1 + n) a^{2n}

⇒ p = h^2 u^{2n+3}(1+ n) a^2n

= h^2 $a^{2n}\frac{1}{r^{2n+3}}$(1 + n)

i.e) p α $\frac{1}{r^{2n+3}}$

⇒ central acceleration varies inversely as the (2n+3)rd pavers of the distance

2. **Find the law of force to an internal point under which a body will describe a circle.**

Solution:

The pedal equation of the circle is c^2=r^2+a^2 -2ap

Difference with respect to r

O = 2r -2a $\frac{dp}{dr}$

⇒ $\frac{dp}{dr}$ = $\frac{r}{a}$

P = $\frac{h^2}{p^3}\frac{dp}{dr}$= $\frac{h^2}{p^3}\left(\frac{r}{a}\right)$

= $\frac{8h^2a^2.r}{(r^2+a^2-c^2)^3}$

3. **A particle moves in an ellipse under a force which is always directed towards its focus. Find the law of force the velocity at any point of the path and its periodic time**

Solution:

The polar equation to the ellipse is $\frac{l}{r}$ = 1+ecos θ …………………①

Where e is the eccentricity and λ is the semilatus rectum

Put u = 1/r

① = >lu = I + ecosθ

Difference with respect to θ

$\lambda \frac{du}{d\theta}$ = -e sin θ

$\frac{du}{d\theta} = \frac{-e}{\lambda}$ sin θ

$\frac{d^2u}{d\theta^2} = \frac{-e}{\lambda}$ cos θ

The equation of central orbit is

$\frac{p}{h^2u^2} = u + \frac{d^2u}{d\theta^2}$

$= u - \frac{e}{\lambda}\cos\theta$

$= \frac{1}{\lambda} + \frac{e}{\lambda}\ cos\theta - \frac{e}{\lambda}\ cos\theta$

$\frac{p}{h^2u^2} = \frac{1}{\lambda}$

$\Rightarrow p = \frac{h^2u^2}{\lambda} = \frac{h^2}{\lambda r^2}$s

P $\alpha \frac{1}{r^2}$

∴ The focus varies inversely as the square of the distance from the pole.